Some Results on Simplicial Homology Groups of 2D Digital Images

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Abstract

In this paper we study some results related to the simplicial homology groups of 2D digital images. We show that if a bounded digital image $X \subset Z$ is nonempty and κ-connected, then its homology groups at the first dimension are a trivial group. In general, we prove that the homology groups of the operands of a wedge of digital images need not be additive.

Keywords

Digital Image; Digital Topology; Homotopy; Simplicial Homology

Introduction

Digital image or picture processing is concerned to a great extent with the extraction and description of objects or regions in pictures, such as individual characters in text, components in circuit diagrams, cells in Pap smears, tumors in x-rays, buildings in aerial photographs, etc. (see [29]).

Digital Topology, introduced in [27], plays an important role in analyzing digital images arising in computer graphics as well as many areas of science including neuroscience, medical imaging, industrial inspections, geoscience, and fluid dynamics. Concepts and results of digital topology are used to specify and justify some important low-level image processing algorithms including algorithms for thinning, boundary extraction, object counting, and contour filling.

Many researchers wish to characterize the properties of a digital image. Therefore it is desirable to have tools from topology (including algebraic topology). Digital versions of some concepts from algebraic topology have been studied in papers [1,2,5,6,9,10,16,15].

Algebraic Topology is concerned with finding algebraic invariants of topological spaces. A powerful topological invariant is homology which characterizes an object by its "p-dimensional holes". Intuitively, the 0-dimensional holes can be seen as connected components, 1-dimensional holes can be seen as tunnels, and 2-dimensional holes can be seen as cavities. Another important property of homology is that local calculations induce global properties. In other words, homology is a tool to study spaces and has been applied in image processing for 2D and 3D images [1].

There have been a few attempts to address homology computation in imaging [21], [23]. More recently, [35], [19] provide a number of useful algorithms for homology but their applicability image analysis remains limited. In [12] homology is used image matching. In the context of Digital Topology the issue is addressed in [15], [24], [16], [26]. In [4] computational homology project works with command prompt. In [32] the work of Saveliev was inspired primarily by the notion of "persistent homology" [14] and grew from the desire to apply this concept to digital image analysis. S. Peltier et al [25] have presented and implemented a technique to compute the whole homology of arbitrary finite shape. They have addressed the problem of extracting generators of the homology groups with a modulo. They have also proposed some computing optimizations, using both Dumas and Agoton algorithms, together with moduli operations. In [26] S. Peltier et al. introduce a method for computing homology groups and their generators of a 2D image, using a hierarchical structure i.e. irregular graph pyramid.

Kaczynski et al. [18] have computed homology groups with a sequence of reductions. The ides is to derive a new object with less cells while preserving homology at each step of the transformation. During the computations, to ensure invertible coefficients,

Kaczynski et al. choose them in a field.

Chen and Rong [11] have designed linear time algorithms to recognize and determine topological invariants such as the genus and homology groups in 3D. These properties can be used to identify patterns in 3D image recognition. [2], [16], [15], [19] have studied simplicial homology groups of digital images. In [10] Boxer at al. build on [2] to expand the knowledge of the simplicial homology groups of digital images.

Havana et al. [2] have studied simplicial holomolgy groups of digital imaes. In particular they compute the homology groups of a minimal simple 18-surface in dimension 3. Boxer and Karaca [10] build on [2] to expand our knowledge of the simplicial homology groups of digital images.

Karaca and Ege [20] present the digital cubical homology groups of digital images which are based on the cubical homology groups of topological spaces in algebraic topology. They investigate some fundamental properties of cubical homology groups of digital images. They also give a relation between digital simplicial homology groups and digital cubical homology groups. They show that the Mayer-Vietoris Theorem need not be hold in digital images.

The purpose of this paper is to give complete algebraic presentation of simplicial homology groups of any objects in a 2-dimensional digital image. Some results related to homology groups of a digital image are given.

Preliminaries

Let Z be the set of integers. Then Z^n is the set of lattice points in the n-dimensional Euclidean space. A (binary) digital image is a finite subset of Z^n with an adjacency relation.

For a positive integer l with $1 \le l \le n$ and distinct two points

$$p = (p_1, p_2, ..., p_n), q = (q_1, q_2, ..., q_n) \in Z^n \quad (2.1)$$

p and q are c_l - adjacent [5] if

- there are at most l indices i such that $|p_i - q_i| = 1$ and
- for all other indices j such that $|p_j - q_j| \neq 1,\ p_j = q_j$.

It is common to denote an adjacency relation κ by the number of κ-adjacent points of a given point. For example,

in Z we have $c_1 = 2$; in Z^2 we have $c_1 = 4$ and $c_2 = 8$; in Z^3 we have $c_1 = 6$, $c_2 = 18$, and $c_3 = 26$. More general adjacency relations are studied in [17]. A κ-neighbor of a lattice point p is κ-adjacent to p.

Let κ be an adjacency relation defined on Z^n. A digital image $X \subset Z^n$ is κ-connected [17] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0,1,..., r-1$. A κ-component of a digital image X is a maximal κ-connected subset of X.

Let $a, b \in Z$ with $a < b$. A set of the form

$$[a,b]_Z = \{z \in Z \mid a \le z \le b\} \quad (2.2)$$

is called a digital interval.

Let $X \in Z^{n_0}$ and $Y \in Z^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency respectively. Two (κ_0, κ_1)-continuous functions $f, g : X \to Y$ are said to be digitally (κ_0, κ_1)-homotopic in Y [5] if there is a positive integer m and a function $H : X \times [0,m]_Z \to Y$ such that

- for all $x \in X$, $H(x,0) = f(x)$ and $H(x,m) = g(x)$,
- for all $x \in X$, the induced function $H_x : [0,m]_Z \to Y$ defined by

  $H_x(t) = H(x,t)$ for all $t \in [0,m]_Z$,

is $(2, \kappa_1)$- continuous, and

- for all $t \in [0,m]_Z$, the induced function $H_t : X \to Y$ defined by

$$H_t(x) = H(x,t) \qquad \text{for all } x \in X,$$

is (κ_0, κ_1)- continuous.

The function H is called a digital (κ_0, κ_1)-homotopy between f and g.

Let $X \in Z^{n_0}$ and $Y \in Z^{n_1}$ be digital images with κ_0 -adjacency and κ_1 -adjacency respectively. A function $f : X \to Y$ is a (κ_0, κ_1)-isomorphism [5] if f is (κ_0 , κ_1)-continuous and bijective and further $f^{-1} : X \to Y$ is (κ_1, κ_0)-continuous, in which case we write $X \approx_{(\kappa_0,\kappa_1)} Y$.

By a digital κ-path from x to y in a digital image X, we mean a (2, κ)-continuous function $f : [0,m]_Z \to X$ such that f(0) = x and f(m) = y. If f(0) = f(m) then the κ-path is said to be closed. A simple closed κ-curve of $m \geq 4$ points in a digital image X is a sequence $\{f(0), f(1),..., f(m-1)\}$ of images of the κ-path $f : [0,m-1]_Z \to X$ such that f(i) and f(j) are κ-adjacent if and only if j = i $\pm$ 1 mod m.

Let S be a set of nonempty subset of a digital image (X, κ). Then the members of S are called simplexes of (X, κ) [33] if the following hold:

a) If p and q are distinct points of $s \in S$, then p and q are κ-adjacent.
b) If $s \in S$ and $\varnothing \neq t \subset s$, then $t \in S$ (note this implies every point p that belongs to a simplex determines a simplex $\{p\}$).

An m-simplex is a simplex S such that $|S| = m+1$.

Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P.

Let (X, κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. If the following statements hold then (X, κ) is called a finite digital simplicial complex [2]:

(1) If P belongs to X, then every face of P also belongs to X.

(2) If P,Q $\in$ X, then P $\cap$ Q is either empty or a common face of P and Q.

The dimension of a digital simplicial complex X is the largest integer m such that X has an m-simplex.

$C_q^\kappa(X)$ can be defined a free abelian group with basis all digital (κ,q)-simplices in X (see [2]).

In [2] it is shown that if (X, κ) $\subset Z^n$ is a digital simplicial complex of dimension m, then for all q > m, $C_q^\kappa(X)$ is a trivial group.

Let (X, κ) $\subset Z^n$ be a digital simplicial complex of dimension m. The homomorphism $\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$ defined by

$$\partial_q(< p_0,..., p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0,..., \hat{p}_i,..., p_q >, q \leq m \\ 0, q > m. \end{cases}$$

is called a boundary homomorphism (where $\hat{p}_i$ means delete the point p_i) [2].

For all $1 \leq q \leq m$, we immediately have $\partial_{q-1} \circ \partial_q = 0$, (see [2]).

Definition 2.1 [2] Let (X, κ) be a digital simplicial complex.

(1) $Z_q^\kappa(X) = Ker\partial_q$ is called the group of digital simplicial q-cycles.

(2) $B_q^\kappa(X) = \operatorname{Im}\partial_{q+1}$ is called the group of digital simplicial q-boundaries.

(3) $H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X)$ is called the qth digital simplicial homology group.

Theorem 2.2 [10] Let (X, κ) be a directed digital simplicial complex of dimension m.

(1) $H_q^\kappa(X)$ is a finitely generated abelian group for every

$q \geq 0$.

(2) $H_q^\kappa(X)$ is a trivial group for all $q > m$.

(3) $H_m^\kappa(X)$ is a free abelian group, possibly zero.

In [10], it is proven that for each $q \geq 0$, $H_q^\kappa(X)$ is a covariant functor from the category of digital simplicial

complexes and simplicial maps to the category of abelian groups.

Main Results

In this section we give some results related to the simplicial homology groups of 2D digital images.

Proposition 3.1 If $X \subset Z^2$ is a κ-connected digital image, then $H_0^{\kappa}(X, Z_2) \cong Z_2$.

Proof. Let $x, y \in X$. Since X is κ-connected, there is a sequence $\{x_i\}_{i=0}^{m} \subset X$ such that $x = x_0, x_m = y$, and $0 < i \le m$ implies x_{i-1} and x_i are κ-adjacent. The latter

İmplies that $\hat{x}_{i+1} + \hat{x}_i \in B_0^{\kappa}(X, Z_2)$. Since

$Z_0^{\kappa}(X, Z_2) = C_0^{\kappa}(X, Z_2)$, we have

$[\hat{x}_{i-1}] = [\hat{x}_{i-1}] + [\hat{x}_{i-1} + \hat{x}_i] = [\hat{x}_i] \in H_0^{\kappa}(X, Z_2)$

for $0 < i \le m$. Thus, $[\hat{x}_i] = [\hat{x}_0]$ for $0 < i \le m$. It follows that $H_0^{\kappa}(X, Z_2) = \{0, [\hat{x}_0]\}$. Since clearly $[\hat{x}_0] \ne 0$, the assertion follows.

We immediately have the following.

Corollary 3.2 Let $X \subset Z^2$ have c distinct κ-components.

Then we have $H_0^{\kappa}(X, Z_2) \cong \bigoplus_{i=1}^{c} Z_2$.

Theorem 3.3 If a bounded digital image $X \subset Z$ (we regard $Z = Z \times \{0\} \subset Z^2$) is nonempty and κ-connected, then $H_1^{\kappa}(X, Z_2)$ is a trivial group.

Proof. By hypothesis, there are integers a and $p > 0$ such that $X = [a, a+p-1]_Z$. If $p = 1$, $Z_1^{\kappa}(X, Z_2) = C_1^{\kappa}(X, Z_2) = \{0\}$, and the assertion follows immediately. Otherwise, $p > 1$, and we have edges $e_i = \{v_i, v_{i+1}\}$ for $0 \le i < p-1$, where $v_i = a + i$ for $0 \le i < p-1$. Then for any element c in $C_1^{\kappa}(X, Z_2)$, we can write

$$\partial_1(c) = \partial_1\left(\sum_{i=0}^{p-2} \alpha_i \hat{e}_i\right) = \sum_{i=0}^{p-2} \alpha_i \partial_1(\hat{e}_i) = \sum_{i=0}^{p-2} \alpha_i (\hat{v}_i + \hat{v}_{i+1}).$$

If $\partial_1(c) = 0$, then we get $\sum_{i=0}^{p-2} \alpha_i (\hat{v}_i + \hat{v}_{i+1}) = 0$. So we can conclude that $\alpha_i = 0$ for $i = 0, 1, \ldots, p-2$. Therefore, we have $Z_1^{\kappa}(X, Z_2) \cong \{0\}$, which gives the desired result. ◎

The next result gives a simple characterization of the 1-cycles of a digital image in Z^2.

Theorem 3.4 Let $X \subset Z^2$ be a digital image, $\kappa \in \{4, 8\}$.

Let $c = \sum_{i=0}^{m} \hat{e}_i \in C_1^{k}(X, Z_2)$, where $i \ne j$ implies $\hat{e}_i \ne \hat{e}_j$. Let $V = \{v_j\}_{j=0}^{p}$ be the set of distinct vertices of the edges in the set $E = \{e_i\}_{i=0}^{m}$. Let G_c be the graph with vertex set V and edge set E. Then $c \in Z_1^{\kappa}(X, Z_2)$ if and only if for every $v \in V$, the degree of v in G_c is even.

Proof. It is easily seen that $\partial_1(c) = \sum_{i=0}^{p} \deg(v_i) \hat{v}_i$, where $\deg(v_i)$ is the degree of v_i in G_c. Since $\hat{v}_i$ are distinct, the assertion follows.

Corollary 3.5 Let $S \subset Z^2$ be a simple closed κ-curve, $\kappa \in \{4, 8\}$. Then $H_1^{\kappa}(S, Z_2) \cong Z_2$.

Proof. The only 1-chain of S that has no vertices with odd degree in the derived graph is the chain whose union of edges is S. From Theorem 3.4, this chain is the only 1-cycle of S. It follows that $H_1^{\kappa}(S, Z_2) \cong Z_1^{\kappa}(S, Z_2) \cong Z_2$. ◎

Definition 3.6 [9] Let X and Y be digital images in Z^n, using the same adjacency notion, denoted by κ, such that $X \cap Y = \{x_0\}$, where x_0 is the only point of X

adjacent to any point of Y and x_0 is the only point of Y adjacent to any point of X. Then the wedge of X and Y, denoted $X \wedge Y$, is the image $X \wedge Y = X \cup Y$, with κ adjacency.

In general, the homology groups of the operands of a wedge of digital images need not be additive – that is, $H_r^{\kappa}(X \wedge Y, Z_2)$ need not be isomorphic to $H_r^{\kappa}(X, Z_2) \oplus H_r^{\kappa}(Y, Z_2)$. For example, if X and Y are κ-connected digital images in Z^2 such that $X \wedge Y$ is defined, then $X \wedge Y$ is also κ-connected, so by Proposition 3.1 we have $H_0^{\kappa}(X \wedge Y, Z_2) \cong Z_2$ and $H_0^{\kappa}(X, Z_2) \oplus H_0^{\kappa}(Y, Z_2) \cong Z_2 \oplus Z_2$. However, if one of the operands of the wedge operation is a digital simple closed curve, we have the following.

Theorem 3.7 Let $Y = X \wedge S \subset Z^2$ be a digital image with κ-adjacency, $\kappa \in \{4,8\}$, where S is a simple closed κ-curve. Then

$$H_1^{\kappa}(Y, Z_2) \cong H_1^{\kappa}(X, Z_2) \oplus H_1^{\kappa}(S, Z_2)$$
$$\cong H_1^{\kappa}(X, Z_2) \oplus Z_2.$$

Proof. We have seen above that the only 1-cycle of S is the chain z_S generated by the union of all the edges of S. It follows from Theorem 3.4 that if $z_Y \in Z_1^{\kappa}(Y, Z_2)$, then either z_Y has all of the edges of S or none of the edges of S. Thus, there exists $z_X \in Z_1^{\kappa}(Y, Z_2)$, such that either $z_Y = z_X$ or $z_Y = z_X + z_S$. Therefore,

$$Z_1^{\kappa}(Y, Z_2) \cong Z_1^{\kappa}(X, Z_2) \oplus Z_1^{\kappa}(S, Z_2)$$
$$\cong Z_1^{k}(X, Z_2) \oplus Z_2.$$

Since $B_1^{\kappa}(Y, Z_2) \cong B_1^{\kappa}(X, Z_2) \cong B_1^{\kappa}(S, Z_2) \cong \{0\}$, it follows that

$$H_1^{\kappa}(Y, Z_2) \cong H_1^{\kappa}(X, Z_2) \oplus H_1^{\kappa}(S, Z_2)$$
$$\cong H_1^{\kappa}(X, Z_2) \oplus Z_2,$$

the latter isomorphism following from Corollary 3.5.

Conclusion

The purpose of this paper is to give complete algebraic presentation of simplicial homology groups of any objects in a 2-dimensional digital image. We present some results related to the simplicial homology groups of 2D digital images. Future works will focus on Eilenberg-Steenrod axioms for the simplicial homology groups of digital images. We also intend to determine which axioms need not be hold in digital images.

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